Here

$$k^2 = d\left(\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right), \quad \omega_s^2 = \frac{1}{\mu_s}\frac{\partial p_s}{\partial x} - \frac{1}{\mu_1+\mu_2}\left(\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x}\right)$$

The accurate solution will have the form

$$U_s = \frac{1}{\mu_1+\mu_2}\left(\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial x}\right) + \frac{\omega^2}{k_0^2}\left(\frac{\operatorname{ch} ky}{\operatorname{ch} kh} - 1\right) \tag{9}$$

and shows that the effect is largely determined by the intensity of the interaction of the components of the gas, which is characterized by the parameter k_0R (Fig. 2). In the case of weak interaction, when the value of this parameter is small ($k_0R = 0.5, 1$ and 2), the components of the binary mixture behave as though they are independent. It is interesting to note that in the case of strong interaction ($k_0R = \infty$), the flow again acquires the form of Poiseuille flow with the overall viscosity of the components and with the overall gradient.

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Translated by R.C.G.

J. Appl. Maths Mechs, Vol. 55, No. 3, pp. 301-309, 1991 0021-8928/91 $15.00+0.00
Printed in Great Britain

EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS*

V.I. GRYN

Considering steady Hiemenz-Birman flows only, a study is made of flows between porous walls, on the assumption that fluid is injected and extracted at identical rates. It is shown that wherever fluid is being extracted a boundary layer forms at the wall. A class of unsteady two-dimensional flows, more general than Hiemenz-Birman flow, is investigated. In a class of flows generalized Jeffrey-Hamel flow, attention is devoted to flows in a dihedral angle between porous walls when fluid is injected and extracted. A class of steady (unsteady) two-dimensional flows is found, in which flow between coaxial porous cylinders, with fluid injected and extracted at arbitrary rates, is considered. Some exact solutions of the steady- and unsteady-state Navier-Stokes equations are found.

**Prikl. Matem. Mekhan.*, 55, 3, 373-381, 1991

1. Flow between porous walls when there is injection and extraction. The equations governing steady two-dimensional viscid incompressible flow /1/

$$u_x + v_y = 0,\ \ u_y - v_x = \omega,\ \ \Delta\omega = R\,(u\omega_x + v\omega_y) \tag{1.1}$$

have solutions of the following form (Hiemenz-Birman flow /2-4/):

$$v = S\,(y),\ \ u = A(y)\,x + T\,(y),\ \ \omega = B\,(y)\,x + \Omega\,(y) \tag{1.2}$$

Other solutions are obtained from (1.2) by rotating the x, y axes. The functions S, A, T, B and Ω satisfy the system of equations

$$A = -S',\ \ B = A',\ \ \Omega = T' \tag{1.3}$$

$$S^{(\mathrm{IV})} + R\,(S'S'' - SS''') = 0,\ \ T''' + R\,(S''T - ST'') = 0 \tag{1.4}$$

Note that Eqs.(1.4) can be written differently as

$$S''' + R\,(S'^2 - SS'') = C_1,\ \ T'' + R\,(S'T - ST') = C_2 \tag{1.5}$$

Throughout, the letters $C_1, C_2, \ldots$ will denote constants.
Eqs.(1.4) have the obvious solutions

$$S = C_3 y + C_4,\quad T = \int_0^y\int_0^t C_5 \exp\left(\tfrac{1}{2}C_3Rs^2 + C_4Rs\right) ds\,dt + C_6 y + C_7$$

and also solutions of the form

$$S = -6/(Ry),\ \ T = C_3y^2 + C_4y^{-2} + C_5y^{-3}$$

If $C_1 = 0$ the equations for S becomes an Abel equation of the second kind.
The pressure corresponding to solutions of type (1.2) is

$$p\,(x,\ y) = -\tfrac{1}{2}C_1R^{-1}\rho x^2 + C_2R^{-1}\rho x + R^{-1}\rho S' - \tfrac{1}{2}\rho S^2 + C_8$$

where ρ is the density of the liquid and C_1 and C_2 are the constants in (1.5). If $C_1 \neq 0$, the substitution $z = x - x_0$ brings us to the case $C_2 = 0$ /3/.

In connection with flows (1.2) with $C_2 = 0$, studies have been published /3, 4/ of flows between porous walls $y = \pm h$ assuming that fluid is injected or extracted at rates $V_\pm$ (at the wall $y = -h$ one has injection if $V_- > 0$, extraction if $V_- < 0$ and no-slip if $V_- = 0$). Assuming that $V_+ = V_- \equiv V$ in /3, 4/ only the solutions $S \equiv V$, $T \equiv 0$ were found. We shall expand these results by conducting a more complete analysis of flows at $V_- = V_+$. In addition, we shall assume that the wall $y = h$ is moving at a horizontal velocity U. The boundary conditions for Eqs.(1.4) are as follows:

$$S\,(\pm h) = V_\pm,\ \ S'\,(\pm h) = 0,\ \ T\,(-h) = 0,\ \ T\,(h) = U,\ \ \int_{-h}^{h} T\,(y)\,dy = Q \tag{1.6}$$

where the integral condition is an analogue of the discharge of liquid in the section $x = 0$ (this condition was replaced in /3, 4/ by $C_2 = 0$).

Streamlines for the case $U = V_- = 0$, $V_+ < 0$ are shown in Fig.1. The porous walls in Fig.1-5 are represented by dashed lines; the solid diagonally hatched lines in Fig.1 and 4 represent solid walls. The existence of back flow at large R has been observed previously /3/.

Let us consider the case $V_+ = V_- \equiv V > 0$, when problem (1.4), (1.6) can be solved in terms of elementary functions:

$$S\,(y) \equiv V,\ \ T\,(y) = \tau\,(y/h),\ \ \tau\,(\xi) = \tfrac{1}{2}U\,(1 + \xi) + (Q - Uh)\,\times$$
$$[\exp\,(\alpha\xi) - \mathrm{sh}\,(\alpha)\,\xi - \mathrm{ch}\,(\alpha)]\,\{2h\,[\alpha^{-1}\,\mathrm{sh}\,(\alpha) - \mathrm{ch}\,(\alpha)]\}^{-1},\ \alpha = RVh$$

The pressure in this case is

$$p\,(x,\ y) = -\rho V\,\{(Q - Uh)\,[2h^2\,(\mathrm{cth}\,(\alpha) - \alpha^{-1})]^{-1} + \tfrac{1}{2}h^{-1}U\}\,x + \mathrm{const}$$

The following asymptotic estimates hold:

$$T\,(y) = \tfrac{1}{2}h^{-1}\,(Q - Uh)\,(1 + \alpha^{-1})\,(1 + y/h) + \tfrac{1}{2}U\,(1 + y/h) + O\,(\alpha^{-2}) \tag{1.7}$$
$$p\,(x,\ y) = -x\rho V\,\{\tfrac{1}{2}h^{-2}\,(Q - Uh)\,(1 + \alpha^{-1}) + \tfrac{1}{2}h^{-1}U + O\,(\alpha^{-2})\} + \mathrm{const}$$
$$y \in [-h,\ h_1],\ h_1 = h\,(1 - 2\alpha^{-1}\ln\alpha),\ \alpha \gg e$$

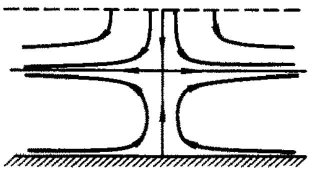

Fig.1

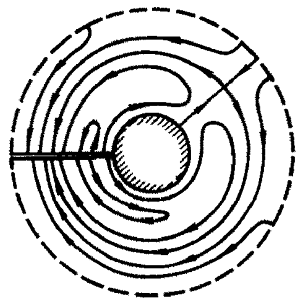

Fig.4

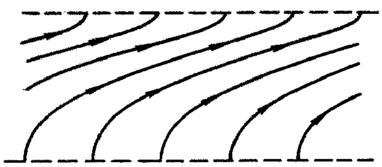

Fig.2

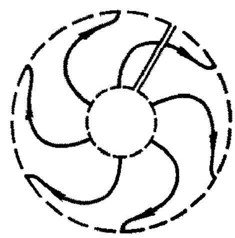

Fig.5

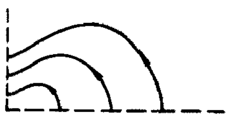

Fig.3

Thus, in this region the solution is identical, to within $O(\alpha^{-2})$, with the solution of Euler's equations and there is no boundary layer at the wall $y=-h$. At the wall $y=h$, however, there is a boundary layer if $Q\neq Uh$. It will be convenient to divide it into two subregions. For $y\in[h_2, h]$, $h_2=h(1-\alpha^{-2})$, we have the following asymptotic estimate:

$$T(y)={}^1/_2h^{-1}(Q-Uh)(2+\alpha^{-1})\,\alpha\,(1-y/h)+{}^1/_2U(1+y/h)+$$
$$O(\alpha^{-2})=U+\alpha(Q/h-U)(1-y/h)+O(\alpha^{-2})$$

The second formula of (1.7) holds for $y\in[-h,h]$, $\alpha\gg e$. Consequently, for $y\in[h_2,h]$ the solution of problem (1.4), (1.6) does not approach a solution of the Euler equations. However, at $y\in[-h,h_1]\cup[h_2,h]$ the streamlines coincide to within $O(\alpha^{-2})$ with the corresponding parabolas. When $y\in[h_1,h_2]$ the parabolas are "spliced" together.

The streamlines

$$x(y;\ C_9)=(Q-Uh)\,[h\alpha^{-1}\exp(h^{-1}\alpha y)-{}^1/_2h^{-1}\,\mathrm{sh}\,(\alpha)\,y^2-\mathrm{ch}\,(\alpha)\,y]\times$$
$$\{2h\,[\alpha^{-1}\,\mathrm{sh}\,(\alpha)-\mathrm{ch}\,(\alpha)]\}^{-1}+{}^1/_2\,U\,(y+{}^1/_2h^{-1}y^2)+C_9$$

at $U=0$ are shown in Fig.2.

2. *Flows allowing transformation to vorticity-stream function variables.* Let $\psi_y=u$, $\psi_x=-v$ and assume that the following hold.

Condition 1. $\psi_y\omega_x-\psi_x\omega_y\neq 0$.

Consider the functions

$$X(\omega,\psi)\equiv\omega_x(x(\omega,\psi),\ y(\omega,\psi)),\quad Y(\omega,\psi)\equiv\omega_y(x(\omega,\psi),\ y(\omega,\psi))$$

Eqs.(1.1) become

$$\frac{\partial G_i}{\partial\omega}+\frac{1}{\sigma}\begin{Vmatrix} f & g\\ -g & f\end{Vmatrix}\frac{\partial G_i}{\partial\psi}=\frac{1}{\sigma}F_i,\quad i=1,2 \tag{2.1}$$
$$\sigma=X^2+Y^2,\quad g=uX+vY,\quad f=uY-vX$$
$$G_1=\begin{Vmatrix}u\\ v\end{Vmatrix},\quad G_2=\begin{Vmatrix}X\\ Y\end{Vmatrix},\quad F_1=\omega\begin{Vmatrix}Y\\ -X\end{Vmatrix},\quad F_2=RgG_2$$

Condition 2. $u=u(\omega)$, $v=v(\omega)$.

If Conditions 1 and 2 both hold, then $dG_i/d\omega=\sigma^{-1}F_i$, $i=1,2$, and therefore $dg/d\omega=\sigma^{-1}Rg^2$, $d\sigma/d\omega=2Rg$. Thus,

$$\sigma = C_1 g^2,\ C_1 > 0,\ g = C_1^{-1} R\omega + C_2,\ X = C_1 C_3 R^{-1} (C_1^{-1} R\omega + C_2) \tag{2.2}$$
$$Y = C_3^{-1} C_4 X,\ C_3^2 + C_4^2 = C_1^{-1} R^2,\ u = C_4 W + C_5,\ v = -C_3 W + C_6,$$
$$W = C_1 R^{-2} (\omega - C_* \ln |\omega + C_*|),\ C_* = C_1 C_2 R^{-1},\ C_3 C_5 + C_4 C_6 = C_1^{-1} R$$

Condition 3. $u = u(ax + by),\ v = v(ax + by),\ a^2 + b^2 = 1.$

Proposition 1. If Conditions 1 and 2 hold, Eqs.(1.1) have exactly the same solutions as when Conditions 1 and 3 hold, and all these solutions are given by (2.2).

Condition 4. $\omega = \omega(ax + by),\ a^2 + b^2 = 1.$

We may assume without loss of generality that $a = 0,\ b = 1$ (rotation of the x, y axes). Let Conditions 1 and 4 hold. Then

$$\omega'' = Rv\omega',\ v = v(y) \neq 0,\ u_x = -v' \equiv g_*(y),\ u_y = \omega$$
$$u = \int_0^y \omega(s)\,ds + f_*(x),\ f_*'(x) = g_*(y) = C_1,\ f_* = C_1 x + C_3$$
$$v = -C_1 y + C_2,\ C_1^2 + C_2^2 \neq 0,\ y \neq C_1^{-1} C_2$$

$$\omega(y) = C_4 \int_0^y \exp(C_2 R s - {}^1/_2 C_1 s^2)\,ds + C_5 \tag{2.3}$$
$$u(x, y) = C_3^* + C_1 x + C_5 y + C_4^* (y - C_1^{-1} C_2) \int_0^y w(s)\,ds + C_1^{-1} C_4 R^{-1} w(y)$$
$$w(y) = \exp[-{}^1/_2 C_1 R (y - C_1^{-1} C_2)^2]$$
$$C_4^* = C_4 \exp({}^1/_2 C_1^{-1} C_2^2 R),\ C_3^* = C_3 - C_1^{-1} C_4 R^{-1}$$

If Conditions 1, 4 are satisfied, all the other conditions are obtained from (2.3) by rotation of the x, y axes.

Condition 5. $X = X(\omega),\ Y = Y(\omega).$

Let Conditions 1 and 5 hold. Then

$$\sigma > 0,\ dG_2/d\omega = \sigma^{-1} F_2,\ g = g(\omega),\ Y dX - X dY = 0 \tag{2.4}$$
$$aX + bY = 0,\ a^2 + b^2 = 1,\ a\omega_x + b\omega_y = 0$$
$$\omega = \omega(ay - bx)$$

The following result follows from (2.3) and (2.4).

Proposition 2. If Conditions 1 and 5 hold, Eqs.(1.1) have exactly the same solutions as when Conditions 1, 4 hold, and all these solutions are either of the form (2.3) or derivable from (2.3) by rotation of the x, y axes.

3. ***Flows in a dihedral angle between porous walls with injection and extraction.*** Let us transform the functions and variables in (1.1) by setting

$$x = r\cos\theta,\ y = r\sin\theta$$
$$u = u_r \cos\theta - u_\theta \sin\theta,\ v = u_r \sin\theta + u_\theta \cos\theta,\ \omega = \omega$$

Then (1.1) becomes

$$\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_\theta}{\partial \theta} = 0,\quad \frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta} + \omega = 0 \tag{3.1}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \omega}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} = R\left[u_r \frac{\partial \omega}{\partial r} + \frac{u_\theta}{r}\frac{\partial \omega}{\partial \theta}\right] \tag{3.2}$$

We will seek solutions on the assumption that Condition 1 holds, which is equivalent to the condition $g \equiv u_r \partial\omega/\partial r + u_\theta r^{-1} \partial\omega/\partial\theta \neq 0$. If $g \equiv 0$ all the local solutions were found in /1/.

Let us assume that the following hold.

Condition 6. $u_\theta = B(r)$.
Then it follows from (3.1) and (3.2) that

$$\frac{4A'(\theta)}{r^4} - \frac{1}{r}\left(r\left(\frac{1}{r}(rB)'\right)'\right)' + \frac{A'''}{r^4} = R\left\{\frac{BA''}{r^3} - \frac{A}{r}\left[\frac{2A'}{r^3} - \frac{1}{r}(rB)'\right]\right\} \tag{3.3}$$

There are two possible cases:

Case 1: $A''(\theta) \not\equiv 0$. Then $u_\theta = C_1/r$.

$$4A' + A''' + 2RAA' - C_1RA'' = 0, \; A'' \not\equiv 0 \tag{3.4}$$

Eq.(3.4) can be transformed to

$$4A + A'' + RA^2 - C_1RA' = C_2, \; A'' \not\equiv 0 \tag{3.5}$$

Eq.(3.5) with $C_1 = 0$ is the Jeffrey-Hamel equation, which occurs in the theory of flows in diverging and converging ducts. A detailed analysis of Eq.(3.5) with $C_1 = 0$ may be found in /5, 6/. We shall expand the description in /5, 6/ by expressing all the solutions in terms of elementary functions when $C_1 = 0$. They are

$$A(\theta) = -C_5^2\,\mathrm{tg}^2\,[(^1/_6R)^{1/2}C_5(\theta + C_4)] - 2R^{-1} - {}^2/_3C_5^2, \quad \theta \in [0,\ 2\pi)$$
$$A(\theta) = C_6^2\,\mathrm{ch}^{-2}\,[(^1/_6R)^{1/2}C_6(\theta + C_4)] - 2R^{-1} - {}^1/_3C_6^2, \quad \theta \in [0,\ 2\pi)$$
$$A(\theta) = -[6(\theta + C_4)^{-2} + 2]/R, \; \theta \in [0,\ 2\pi); \; C_5 \neq 0, \; C_6 \neq 0$$

The last two solutions have a cut along the ray $\theta = 0$ in the x, y plane. Among these solutions there are some satisfying no-slip conditions at the rays $\theta = \pm\theta_*$. They are

$$A(\theta) = 2R^{-1}[3\alpha^2\,\mathrm{ch}^{-2}(\alpha\theta) - (1 + \alpha^2)], \; \alpha^2 \in (^1/_2,\ \infty)$$
$$|\theta| \leqslant \theta_*(\alpha), \quad \theta_*(\alpha) = \ln\left[\left(\frac{3}{\alpha^{-2}+1}\right)^{1/2} - \left(2 - \frac{3}{1+\alpha^2}\right)^{1/2}\right]$$

The no-slip condition holds along the rays $\theta = \pm\theta_*(\alpha)$. The function $\theta_*(\alpha)$ increases monotonically from 0 at $\alpha = 2^{-1/2}$ to $\ln(3^{1/2} + 2^{1/2})$ at $\alpha = \infty$.

Flows with $C_1 \neq 0$ may also be of some interest. In this class we consider steady two-dimensional viscous incompressible flow between two porous walls $\theta = 0$, $\theta = 2\beta$. Fluid is injected (extracted) at the wall $\theta = 0$ $(\theta = 2\beta)$ at a rate $C_1/r \geqslant 0$. The discharge Q of the liquid through the cross-section $r = \mathrm{const}$ is prescribed. The boundary conditions added to Eq.(3.4) are

$$A(0) = A(2\beta) = 0, \quad \rho\int_0^{2\beta} A(\theta)\,d\theta = Q$$

The case $C_1 = 0$ corresponds, if $Q > 0$, to Jeffrey-Hamel flow in a diverging duct, and if $Q < 0$ - in a converging duct. The streamlines in the case $Q < 0, C_1 > 0$ are shown in Fig.3.

If is not our purpose here to analyse the boundary layer at the wall $\theta = 2\beta$. We mention that some of the flows described by Eq.(3.4) have infinite radial velocity along certain rays $(|A(\theta_*)| = \infty)$. Thus, if $C_1 > 0$ flows with closed streamlines may exist.

Thus, in Case 1 the flows have the form

$$u(\theta) = C_1/r, \; u_r = A(\theta)/r, \; p = (2Rr^2)^{-1}\rho\,[4A(\theta) - C_2 - C_1^2R] + C_3 \tag{3.6}$$

where A, C_1 and C_2 are as in (3.5).

Case 2. $A = C_6\theta + C_1$. Then we infer from (3.3) that $C_6 = 0$, and consequently, assuming the truth of Condition 1, we have

$$u_\theta = -[d(d+2)]^{-1}C_2r^{d+1} - {}^1/_2C_3r + C_4r^{-1}, \; d = C_1R \neq -2 \tag{3.7}$$
$$u_\theta = -(dr)^{-1}C_2\ln r - {}^1/_2C_3r + C_4r^{-1}, \; d = -2; \; u_r = C_1r^{-1}$$
$$p = -\frac{C_1^2\rho}{2r^2} + \rho\int_1^r \frac{u_\theta^2(s)}{s}\,ds + C_1C_3\rho\theta + C_5, \quad C_1 \neq 0, \quad C_2 \neq 0$$

If $C_3 \neq 0$ there is a cut along the ray $\theta = 0$.

Proposition 3. If Conditions 1 and 6 hold, all solutions of Eqs.(1.1) have the form (3.6), (3.7).

4. Flows between coaxial porous cylinders with injection and extraction. Let us assume that

Condition 7. $u_r = u_r(r)$.

It follows from (3.1)-(3.3) that

$$u_r = A(r)/r,\ u_\theta = -A'\theta + B(r)/r,\ \omega = r^{-1}[(rA')'\theta - B'] \tag{4.1}$$

$$(r(r^{-1}(rA')')')' + R[r^{-1}A'(rA')' - A(r^{-1}(rA')')'] = 0 \tag{4.2}$$
$$(r(r^{-1}B')')' + R[r^{-2}B(rA')' - A(r^{-1}B')' = 0$$

The pressure is

$$p = -{}^1/_2\rho[(A')^2 - r^{-1}A(rA')' + R^{-1}r(r^{-1}(rA')')']\theta^2 + \rho[A'B - AB' + R^{-1}(rB'' - B')]r^{-1}\theta - \rho\int[r^{-2}AA' - r^{-3}(A^2+B^2) - R^{-1}r^{-2}(rA')']\,dr + C_9$$

If $A' \not\equiv 0$ or $B' \not\equiv C_8(1+RA)r$, there will be a cut at $\theta = \pi$.

Proposition 4. If Condition 7 holds, the solutions of Eqs.(3.1) and (3.2) constitute a seven-parameter family of type (4.1), where the functions A and B satisfy (4.2).

The flows (4.1) are analogous to (1.2). We note that formulae (1.2) exhaust all the flows for which $v = v(y)$.

Integration yields the following family of solutions of Eqs.(4.2):

$$B = \int_1^r C_3 t \int_1^t s^c \exp[{}^1/_2 C_1 R(\ln s)^2]\,ds\,dt + {}^1/_2 C_4 r^2 + C_5$$
$$A = C_1 \ln r + C_2;\ c = C_2 R - 1$$

If $C_1^2 + C_2^2 \neq 0$ and $C_3 \neq 0$, Condition 1 holds (the solution depends on R).

Within the class of flows (1.2), let us consider flow between coaxial porous cylinders $r = r_\pm$, with a fluid being injected or extracted at rates $V_\pm$, $V_+r_+ \neq V_-r_-$. Under these circumstances there is a cut at the ray $\theta = \pi$, acting as a sink for some r values and a source for other r values. The boundary conditions for the set of Eqs.(4.2) are

$$A(r_\pm) = V_\pm r_\pm,\quad A'(r_\pm) = B(r_\pm) = 0,\quad \int_{r_-}^{r_+} \frac{B(r)}{r}\,dr = Q \tag{4.3}$$

where the integral condition is an analogue of the discharge of liquid through the section $\theta = 0$.

The streamlines in the case $V_- = 0, V_+ < 0, Q < 0$ are shown in Fig.4. The two close-lying straight-line segments in Figs.4 and 5 represent cuts. The structure of the flow in Fig.4 is similar to that of the flow in Fig.1.

Let us consider the case

$$V_+r_+ = V_-r_- \equiv \gamma \neq 0,\ \infty > r_+ > r_- > 0 \tag{4.4}$$

The boundary conditions for Eqs.(4.2) now have the form (4.3), (4.4). At $\gamma = 0$ we have the well-known case of Couette flow. Problem (4.2)-(4.4) can be solved in terms of elementary functions:

$$A(r) \equiv \gamma,\quad B(r) = Q\delta^{-1}(\alpha)\left[\chi(r,\alpha) - \frac{\chi(r_+) - \chi(r_-)}{r_+^2 - r_-^2} r^2 - \frac{\chi(r_+)r_+^{-2} - \chi(r_-)r_-^{-2}}{r_+^{-2} - r_-^{-2}}\right]$$
$$\alpha = R\gamma \neq 0,\ \chi(r,\alpha) = r^m,\ \alpha \neq -2,\ \chi(r,-2) = \ln r$$
$$\delta(\alpha) = {}^1/_2 r_-^m[a^m - 1]\nu(a^2, {}^1/_2\alpha),\ \alpha \neq -2,\ m = \alpha + 2$$
$$\nu(z,\beta) = [(z-1)^{-1} - (z^q - 1)^{-1}]\ln z - q^{-1}\beta,\ q = \beta + 1$$
$$\delta(-2) = \ln a[{}^1/_2(\ln a - 1) + (a^2 - 1)^{-1}],\ a = r_+/r_-,\ \chi(r_\pm) = \chi(r_\pm, \alpha)$$

It can be shown that $\delta(\alpha) \neq 0$, and then problem (4.2)-(4.4) has a solution.

We shall prove that $\nu(z,\beta) \neq 0$ if $z > 1,\ \beta \neq 0,\ \beta \neq -1$.

If $z > 1$ we have $\nu(z, 1) < 0,\ \nu(z, -2) > 0$.

Fixing β, we have

$$\nu(z,\beta) = -{}^1/_{12}\beta(z-1)^2 + O((z-1)^3),\ z \to 1$$

For fixed $z > 1$ it is true that

$$\nu(z, \beta) = \beta [z \ln^2(z)(z-1)^{-2} - 1] + O(\beta^2), \ \beta \to 0$$
$$\nu(z, \beta) = \ln(z)(z-1)^{-1} - 1 + \beta^{-1} + O(\beta^{-2}), \ \beta \to +\infty$$
$$\nu(z, \beta) = (z-1)^{-1} z \ln(z) - 1 + \beta^{-1} + O(\beta^{-2}), \ \beta \to -\infty$$
$$\nu(z, \beta) < 0, \ \beta \geqslant [(z-1)\ln^{-1}(z) - 1]^{-1}$$

Now, to prove that $\delta(\alpha) \neq 0$, we need only observe that

$$\partial\nu/\partial\beta = \eta^{-2}[\ln^2(z^\eta)(z^\eta - z^{-\eta})^{-2} - 1/4] < 0, \ \eta \neq 0, \ z > 1, \ \eta = 1/2(\beta+1)$$
$$\delta(-2) \geqslant 1/2 \ln a [1/2 \ln(3 + 8^{1/2}) - (1 + 2^{-1/2})^{-1}] > 0{,}1478 \ln a$$

Thus a solution exists for any R, γ, r_+, r_-.
In accordance with (3.7), the pressure is

$$p(r, \theta) = -\frac{\rho\gamma^2}{2r^2} + \rho \int\limits_{r_-}^{r} \frac{B^2(s)}{s^3} ds + 2\rho\gamma Q \frac{\chi(r_+) - \chi(r_-)}{\delta(\alpha)(r_+^2 - r_-^2)} \theta + C_5$$

If $\gamma Q \neq 0$ there is a cut along the ray $\theta = \pi$.
The following estimates hold for $r \in [r_-, r_*]$, $r_* = r_+ \alpha^{-2/\alpha}$, $\alpha = R\gamma > 0$:

$$B(r) = b(r) + O(\alpha^{-2}), \ b(r) = 1/2 Q a_* r_+^{\alpha+2} [2a_* \ln(a) - 1 + 2/\alpha](1 - r_-^{-2} r^2)$$
$$p(r, \alpha) = -\frac{\rho\gamma^2}{2r^2} + \rho \int\limits_{r_-}^{r} \frac{b^2(s)}{s^3} ds - \rho\gamma b''\theta + C_5 + O(\alpha^{-2}),$$
$$\alpha \to +\infty, a_* = (a^2 - 1)^{-1}$$

Thus, in this region, if $\gamma > 0, R \to \infty$, the solution is identical to within $O(\alpha^{-2})$ with the corresponding solution of Euler's equations. A boundary layer forms in $[r_*, r_+]$. For lack of space we shall not analyse its structure. Similarly, if $\gamma < 0$, $R \to \infty$, a boundary layer forms only at a wall through which fluid is being extracted, provided that $r \in [r_-, r_- |\alpha|^{-2/\alpha}]$. Outside this region the solution is identical to within $O(\alpha^{-2})$ with that of Euler's equations.

Fig.5 illustrates the streamlines for $Q < 0, \gamma > 0$.

5. ***First class of unsteady flows.*** The equations governing unsteady two-dimensional viscid incompressible flow are

$$u_x + v_y = 0, \quad u_y - v_x = \omega, \quad R[\omega_t + u\omega_x + v\omega_y] = \Delta\omega \tag{5.1}$$

Eqs.(5.1) have solutions of the form

$$v = S(y, t), \ u = A(y, t)x + T(y, t), \ \omega = B(y, t)x + \Omega(y, t)$$

The functions S, A, T, B and Ω satisfy the system of equations

$$A = -S_y, \ B = A_y, \ \Omega = T_y \tag{5.2}$$
$$S_{yyy} + R[-S_{yt} - SS_{yy} + (S_y)^2] = g(t), \quad T_{yy} - R[T_t - S_yT + ST_y] = \xi(t)$$

where $g(t)$ and $\xi(t)$ are arbitrary function.

The pressure here is

$$p(x, y, t) = -\frac{\rho g(t) x^2}{2R} + \frac{\rho\xi(t) x}{R} + \frac{\rho S_y}{R} - \frac{\rho S^2}{2} - \int\limits_0^y S_t(s, t) ds + \mu(t)$$

The following families of unsteady solutions of Eqs.(5.2) may be found by quadratures.

First family.

$$S = ay + b, \quad \{a, b\} = \text{const}, \quad a \neq 0, \quad T = \int\limits_{y_0(t)}^{y} \eta(s, t) ds$$
$$\eta = \exp(nat) L_n(ay + b, C_1, C_2), \ n = \pm 1, \pm 2, \ldots$$
$$L_{-n}(\xi, C_1, C_2) = \mathrm{M}_+(\xi)\mathrm{N}_n^+, \ n = 1, 2, \ldots; \ L_{n+1}(\xi, C_1, C_2) = \mathrm{N}_n^-,$$
$$n = 0, 1, \ldots$$
$$\mathrm{N}_n^{\pm} = \frac{d^n}{d\xi^n}\left[\mathrm{M}_{\mp}(\xi)\left(C_1 + C_2 \int\limits_0^{\xi} \mathrm{M}_{\pm}(s) ds\right)\right], \quad \mathrm{M}_{\pm}(\xi) = \exp\left(\pm\frac{R\xi^2}{2a}\right)$$

where $L_{-n}(\xi, (-1)^n, 0)$ are the Chebyshev-Hermite polynomials. The First Family of solutions is derived on the basis of well-known results /7, pp.377, 378/ and using a method due to Zbornik /7, pp.568-570/ (however, not all the solutions exhibited above may be found in /7/).

Second family. If $S = a(t)y + b(t)$, then the function $\eta = Ty$ satisfies the heat conduction equation

$$\eta_t = R^{-1}\eta_{yy} - [a(t)y + b(t)]\eta_y \tag{5.3}$$

Eq.(5.3) has solutions

$$\eta = \exp(\delta(t)y^2 + \beta(t)y)\sum_{k=0}^{n}\alpha_k(t)y^k, \quad n = 0, 1, \ldots \tag{5.4}$$

In this case we obtain a non-linear system of $n+3$ ordinary differential equations in the $n+3$ coefficients δ, β, $\alpha_0, \alpha_1, \ldots, \alpha_n$. Solutions can be obtained by quadratures in the case $\delta = \beta \equiv 0$ and in the case $\alpha_1 = \alpha_2 = \ldots = \alpha_n \equiv 0$. Some of the solutions (5.4) go to infinity in a finite time.

Third family. If $S = b(t)$, the change of variables

$$\tau = t, \quad \xi = R^{1/2}y - R^{1/2}\int_0^t B(s)\,ds$$

converts (5.3) into the heat equation $\eta_\tau = \eta_{\xi\xi}$.

Fourth family. If $S = -6/(Ry)$, the second of Eqs.(5.2) may be converted, by transforming the function and one of the variables

$$U(s, t) = s^3 T(R^{-1/2}s, t), \quad s = R^{1/2}y$$

into the non-homogeneous heat conduction equation

$$U_t - U_{ss} = -R^{-1}s^3\xi(t)$$

Other unsteady solutions may be obtained by rotation of the x, y axes.

6. *Second class of unsteady flows.* The same change of functions and variables as in Sect.3 reduces Eqs.(5.1) to the form of Eqs.(3.1) and the equation

$$-R\frac{\partial\omega}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\omega}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2\omega}{\partial\theta^2} = R\left[u_r\frac{\partial\omega}{\partial r} + \frac{u_\theta}{r}\frac{\partial\omega}{\partial\theta}\right] \tag{6.1}$$

Eqs.(3.1), (6.1) have solutions of the form

$$u_r = \frac{A(t,r)}{r}, \quad u_\theta = -\theta\frac{\partial A}{\partial r} + \frac{B(t,r)}{r}, \quad \omega = \frac{\theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) - \frac{1}{r}\frac{\partial B}{\partial r},$$
$$\theta \in [0, 2\pi)$$

The functions A and B satisfy the system of equations

$$-\frac{\partial^2}{\partial t\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{R}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)\right)\right) + \left[\frac{1}{r}\frac{\partial A}{\partial r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) - A\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)\right)\right] = 0 \tag{6.2}$$

$$-\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial B}{\partial r}\right) + \frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial B}{\partial r}\right)\right) + \left[\frac{B}{r^3}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) - \frac{A}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial B}{\partial r}\right)\right] = 0 \tag{6.3}$$

The pressure in this case is

$$p(t,r,\theta) = -\frac{\rho}{2}\left[-r\frac{\partial^2 A}{\partial t\partial r} + \left(\frac{\partial A}{\partial r}\right)^2 - \frac{A}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{r}{R}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)\right)\right]\theta^2 + \left[-r\frac{\partial B}{\partial t} + B\frac{\partial A}{\partial r} - A\frac{\partial B}{\partial r} + \frac{1}{R}\left(r\frac{\partial^2 B}{\partial r^2} - \frac{\partial B}{\partial r}\right)\right]\frac{\rho\theta}{r} - \rho\int\left[\frac{1}{r}\frac{\partial A}{\partial t} + \frac{A}{r^2}\frac{\partial A}{\partial r} - \frac{A^2+B^2}{r^3} - (Rr^2)^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)\right]dr + \Phi(t), \quad \rho = \text{const} > 0$$

where $\Phi(t)$ is an arbitrary function.

Particular solutions of Eq.(6.2) are the functions $A = D(t)\ln r + E(t)$. In that case

Eq.(6.3) becomes the heat conduction equation

$$U_t - R^{-1}U_{rr} + [R^{-1} + E(t) + D(t)\ln r](r^{-1}U_r - r^2U) = 0,\quad U = B_r \tag{6.4}$$

Thus, if $D \equiv 0$, $E \equiv -1/R$ all bounded solutions of Eq.(6.4) may be found by using Fourier transforms. One such solution is

$$u_r = -(Rr)^{-1},\quad u_\theta = \exp(-R^{-1}\pi^2 t)(\pi r)^{-1}\sin(\pi r)$$
$$\omega = -\exp(-R^{-1}\pi^2 t)\, r^{-1}\cos(\pi r)$$
$$p(r,t) = -\rho(2R^2r^2)^{-1} + \rho\pi^{-2}\exp(-R^{-1}\pi^2 t)\int_1^r s^{-3}\sin^2(\pi s)\,ds + \Phi(t)$$

This represents unsteady viscous incompressible flow between the two porous cylinders $r_1 = 1$, $r_2 = n$ with injection and extraction, where $n = 2, 3, \ldots$

I wish to thank Yu.D. Shmyglevskii for his interest and O.V. Troshkin for useful comments.

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Translated by D.L.